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The masses of the light quarks

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Abstract

The talk reviews the current status of knowledge concerning m_u, m_d and m_s. Qualitative aspects of the resulting picture for the breaking of isospin and eightfold way symmetries are discussed. At a more quantitative level, the review focuses on the chiral perturbation theory results for the masses of the Goldstone bosons. The corresponding bounds and estimates for the ratios m_u/m_d and m_s/m_d are described in some detail.

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The first crude estimates [1] for the magnitude of the three lightest quark masses appeared 20 years ago:

$$\begin{array}{llll} m_u \simeq 4\,\mathrm{MeV} & m_d \simeq 6\,\mathrm{MeV} & m_s \simeq 135\,\mathrm{MeV} & [2]\ , \\ m_u \simeq 4.2\,\mathrm{MeV} & m_d \simeq 7.5\,\mathrm{MeV} & m_s \simeq 150\,\mathrm{MeV} & [3]\ . \end{array}$$

Many papers dealing with the pattern of quark masses have appeared since then. I wish to review the current status of our knowledge in this regard. The punch line reads: The numbers have barely changed. The text consists of two parts: a very short first section, dealing with the absolute magnitude of m_s, and a lengthy remainder, concerning the relative size of the three masses, characterized by the ratios m_u/m_d and m_s/m_d.

1 Magnitude of m_s

The best determinations of the magnitude of m_s rely on QCD sum rules [4]. A detailed discussion of the method in application to the mass spectrum of the quarks was given in 1982 [1]. The result for the $\overline{\mathrm{MS}}$ running mass at scale $\mu = 1\,\mathrm{GeV}$ quoted in that report is $m_s = 175 \pm 55\,\mathrm{MeV}$. The issue has been investigated in considerable detail since then [5]. The value reported in the most recent paper [6],

$$m_s = 175 \pm 25\,\mathrm{MeV}\ , \qquad (1)$$

summarizes the state of the art: the central value is confirmed and the error bar is reduced by about a factor of two. The residual uncertainty mainly reflects the systematic errors of the method, which it is difficult to narrow down further.

There is considerable progress in the numerical simulation of QCD on a lattice [7]. For gluodynamics and bound states of heavy quarks, this approach already yields significant results. The values obtained for m_s are somewhat smaller than the one given above. The APE collaboration [8], for instance, reports $m_s = 128 \pm 18\,\mathrm{MeV}$ for the $\overline{\mathrm{MS}}$ running mass at $\mu = 2\,\mathrm{GeV}$. It is difficult, however, to properly account for the vacuum fluctuations generated by quarks with small masses. Further progress with light dynamical fermions is required before the numbers obtained for m_u, m_d or m_s can be taken at face value. In the long run, however, this method will allow an accurate determination of all of the quark masses.

2 Mass spectrum of the pseudoscalars

The best determinations of the *relative* size of m_u, m_d and m_s rely on the fact that these masses happen to be small, so that the properties of the theory may be analysed by treating the quark mass term in the Hamiltonian of QCD as a perturbation. The Hamiltonian is split into two pieces:

$$H_{\rm QCD} = H_0 + H_1 \ \ ,$$

where H_0 describes the three lightest quarks as massless and H_1 is the corresponding mass term,

$$H_1 = \int d^3x \{ m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s \} \ \ .$$

H_0 is invariant under the group $\mathrm{SU}(3)_R \times \mathrm{SU}(3)_L$ of independent flavour rotations of the right- and lefthanded quark fields. The symmetry is broken spontaneously: the eigenstate of H_0 with the lowest eigenvalue, $|0\rangle$, is invariant only under the subgroup $\mathrm{SU}(3)_V \subset \mathrm{SU}(3)_R \times \mathrm{SU}(3)_L$. Accordingly, the spectrum of H_0 contains eight Goldstone bosons, $\pi^\pm, \pi^0, K^\pm, K^0, \overline{K}^0, \eta$. The remaining levels form degenerate multiplets of $\mathrm{SU}(3)_V$ of non-zero mass.

The perturbation H_1 breaks the symmetry, because it connects the right- and left-handed components: $\bar{u}u = \bar{u}_R u_L + h.c.$ In so far as the quark masses m_u, m_d, m_s are small, the entire term H_1 represents a small perturbation, so that the group $\mathrm{SU}(3)_R \times \mathrm{SU}(3)_L$ still represents an *approximate* symmetry of the full Hamiltonian. The perturbation splits the SU(3) multiplets, in particular also the Goldstone boson octet. To first order in the perturbation, the square of the pion mass is given by the expectation value of the perturbation, $M^2_{\pi^+} = \langle \pi^+ | \, m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s | \pi^+ \rangle$ and is therefore linear in the quark masses. Since the eigenstates entering here are those of H_0, the matrix elements respect $\mathrm{SU}(3)_V$ symmetry. In particular, isospin conservation requires $\langle \pi^+ | \bar{u}u | \pi^+ \rangle = \langle \pi^+ | \, \bar{d}d | \pi^+ \rangle \equiv B$. Moreover, since the subgroup $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L$ becomes an exact symmetry for $m_u, m_d \rightarrow 0, m_s \neq 0$, the pion mass must disappear in this limit, so that the matrix element $\langle \pi^+ | \bar{s}s | \pi^+ \rangle$ must vanish. The expansion of $M^2_{\pi^+}$ in powers of m_u, m_d, m_s thus starts with

$$M^2_{\pi^+} = (m_u + m_d) B \{ 1 + O(m_u, m_d, m_s) \} \ \ . \qquad (2)$$

The operation $d \rightarrow s$ takes the π^+ into the K^+, and the K^0 may be reached from there with $u \rightarrow d$. Hence the corresponding lowest order mass formulae read

$$M_{K^+}^2 = (m_u + m_s)B + \dots \quad , \qquad M_{K^0}^2 = (m_d + m_s)B + \dots \tag{3}$$

In the ratios $M_{\pi^+}^2 : M_{K^+}^2 : M_{K^0}^2$, the constant B drops out. Using the Dashen theorem [9] to account for the e.m. self energies, these relations imply [3]

$$\begin{aligned} \frac{m_u}{m_d} &= \frac{M_{K^+}^2 - M_{K^0}^2 + 2M_{\pi^0}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} \quad , \\ \frac{m_s}{m_d} &= \frac{M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} \quad . \end{aligned} \tag{4}$$

Numerically, this gives

$$\frac{m_u}{m_d} = 0.55 \quad , \qquad \frac{m_s}{m_d} = 20.1 \quad . \tag{5}$$

3 Approximate flavour symmetries

The most striking aspect of this result is that the three masses are very different. In particular, the value for m_u/m_d shows that the masses of the u- and d-quarks are quite different. This appears to be in conflict with the oldest and best established internal symmetry of particle physics, isospin. Since u and d form an $I = \frac{1}{2}$ multiplet, isospin is a symmetry of the QCD Hamiltonian only if $m_u = m_d$.

The resolution of the paradox is that m_u, m_d are very small. Disregarding the e.m. interaction, the strength of isospin breaking is determined by the magnitude of $|m_u - m_d|$, not by the relative size m_u/m_d. The fact that m_d is larger than m_u by a few MeV implies, for instance, that the neutron is heavier than the proton by a few MeV. Compared with the mass of the proton, this amounts to a fraction of a per cent. In the case of the kaons, the relative mass splitting $(M_{K^0}^2 - M_{K^+}^2)/M_{K^+}^2$ is more important, because the denominator is smaller here: the effect is of order $(m_d - m_u)/(m_u + m_s) \simeq 0.02$, but this is still a small number. One might think that for the pions, where the square of the mass is proportional to $m_u + m_d$, the relative mass splitting should be large, of order $(M_{\pi^0}^2 - M_{\pi^+}^2)/M_{\pi^+}^2 \propto (m_d - m_u)/(m_d + m_u) \simeq 0.3$, in flat contradiction with observation. It so happens, however, that the pion matrix elements of the isospin breaking part of the Hamiltonian, $\frac{1}{2}(m_u - m_d)(\bar{u}u - \bar{d}d)$, vanish because the group SU(2) does not have a d-symbol. This implies that the mass difference between π^0 and π^+ is of second order in $m_d - m_u$ and therefore tiny. The observed mass difference is almost exclusively due to the electromagnetic self energy of the π^+. So, the above quark mass pattern is perfectly consistent with the fact that the isospin is an almost exact symmetry of the strong interaction: the matrix elements of the term $\frac{1}{2}(m_u - m_d)(\bar{u}u - \bar{d}d)$ are very small compared with those of H_0. In particular, the pions are protected from isospin breaking.

QCD also explains another puzzle: Apparently, the mass splittings in the pseudoscalar octet are in conflict with the claim that SU(3) represents a decent approximate symmetry. This seems to require $M_K^2 \simeq M_\pi^2$, while experimentally, $M_K^2 \simeq 13 M_\pi^2$. The above first order mass formulae yield $M_K^2/M_\pi^2 = (\hat{m}+m_s)/(m_u+m_d)$, where $\hat{m} = \frac{1}{2}(m_u+m_d)$ is the mean mass of u and d. The kaons are much heavier than the pions, because it so happens that m_s is much larger than $\hat{m}$. For SU(3) to be a decent approximate symmetry, it is not necessary that the difference $m_s - \hat{m}$ is small with respect to the sum $m_s + \hat{m}$, because the latter does not represent the relevant mass scale to compare the symmetry breaking with. If the quark masses were of the same order of magnitude as the electron mass, SU(3) would be an essentially perfect symmetry of QCD; even in that world $m_s \gg \hat{m}$ implies that the ratio M_K^2/M_π^2 strongly differs from 1. The strength of SU(3) breaking does not manifest itself in the mass ratios of the pseudoscalars, but in the symmetry relations between the matrix elements of the operators $\bar{u}u$, $\bar{d}d$, $\bar{s}s$, which are used in the derivation of the above mass formulae. The asymmetries in these are analogous to the one seen in the matrix elements of the axial vector currents, $F_K/F_\pi = 1.22$, which represents an SU(3) breaking of typical size. The deviation from the lowest order mass formula,

$$\frac{M_K^2}{M_\pi^2} = \frac{\hat{m}+m_s}{m_u+m_d}\{1+\Delta_M\} \ ,$$

is expected to be of the same order of magnitude, $1+\Delta_M \leftrightarrow F_K/F_\pi$.

The Gell-Mann-Okubo formula yields a good check. The lowest order mass formula for the η reads

$$M_\eta^2 = \tfrac{1}{3}(m_u + m_d + 4m_s)B + \ldots \ , \tag{6}$$

so that the mass relations for π, K, η lead to $3M_\eta^2 + M_\pi^2 - 4M_K^2 = 0$. The accuracy within which this consequence of SU(3) symmetry holds is best seen by working out the quark mass ratio $m_s/\hat{m}$ in two independent ways: while the mass formulae for K and π imply $m_s/\hat{m}=(2M_K^2-M_\pi^2)/M_\pi^2=25.9$, those for η and π yield $m_s/\hat{m}=\frac{1}{2}(3M_\eta^2-M_\pi^2)/M_\pi^2=24.2$. Despite $M_K^2/M_\pi^2 \simeq 13$, the mass pattern of the pseudoscalar octet is a showcase for the claim that SU(3) represents a decent approximate symmetry of QCD.

4 Generalized chiral perturbation theory

I add a few remarks concerning an alternative scenario, called generalized chiral perturbation theory [10]. The scenario may be motivated by an analogy with spontaneous magnetization. There, spontaneous symmetry breakdown occurs in two quite different modes: ferromagnets and antiferromagnets. For the former, the magnetization develops a non-zero expectation value, while for the latter, this does not happen. In either case, the symmetry is spontaneously broken (for a discussion of the phenomenon within the effective Lagrangian framework, see [11]). The example illustrates that operators which are allowed by the symmetry to pick up an expectation value may, but need not, do so.

The standard low energy analysis assumes that the quark condensate is the leading order parameter of the spontaneously broken symmetry. The relation of Gell-Mann, Oakes and Renner [12] states that the leading term in the expansion $M_\pi^2=(m_u+m_d)B\{1+O(m_u,m_d,m_s)\}$ is determined by this parameter: $B=|\langle 0|\bar{u}u|0\rangle|/F_\pi^2$. The simplest version of the question addressed in the papers quoted above is whether the quark condensate indeed tends to a non-zero limit when the quark masses are turned off, like the magnetization of a ferromagnet, or whether it tends to zero, as is the case for the magnetization of an antiferromagnet. If the second option were realized in nature, the pion mass would not be determined by the quark condensate, but by the terms of order m^2, which the Gell-Mann-Oakes-Renner formula neglects. More generally, one may envisage a situation where the quark condensate is different from zero, but small, such that, at the physical value of the quark masses, the terms of order m and m^2 both yield a significant contribution. As pointed out in ref. [10], the available direct experimental evidence does not exclude this possibility. The problem is related to the ambiguity discussed below, in section 7.

The generalized scenario is covered by the standard effective Lagrangian, but allows some of the effective coupling constants at order p^4 to take very large values (l_3, L_7, L_8). To avoid large corrections from these, the standard perturbation series must then be reordered. The main problem with this is that much of the predictive power of the standard framework is lost. The most prominent example is the Gell-Mann-Okubo formula, which does not follow within that scenario. Quite irrespective, however, of whether or not the scheme is theoretically attractive, it is important to subject the issue to experimental test. In this connection, I wish to draw the reader's attention to the beautiful experimental proposal of Nemenov and co-workers [13], who aim at observing $\pi^+\pi^-$ atoms. These decay into a pair of neutral pions, through the strong transition $\pi^+\pi^- \rightarrow \pi^0\pi^0$. Since the momentum transfer nearly vanishes, the decay rate is determined by the combination $a_0 - a_2$ of S-wave $\pi\pi$ scattering lengths. Now, chiral symmetry implies that Goldstone bosons of zero energy do not interact. Hence a_0, a_2 vanish in the limit $m_u, m_d \rightarrow 0$. In other words, the transition amplitude directly measures the symmetry breaking generated by m_u, m_d. Standard chiral perturbation theory yields very sharp predictions for a_0, a_2 [14], while the generalized scenario does not [15]. A measurement of the lifetime of a $\pi^+\pi^-$ atom would thus allow us to decide whether or not the quark condensate represents the leading order parameter.

5 Mass formulae to second order

The leading order mass formulae are subject to corrections arising from contributions which are of second or higher order in the perturbation H_1. A systematic method for the analysis of the higher order contributions is provided by the effective Lagrangian method [16, 17]. In this approach, the quark and gluon fields of QCD are replaced by a set of pseudoscalar fields describing the degrees of freedom of the Goldstone bosons π, K, η. The effective Lagrangian only involves these fields and their derivatives, but contains an infinite string of vertices. For the calculation of the pseudoscalar masses to a given order in the perturbation H_1, however, only a finite subset contributes. The term Δ_M, which describes the SU(3) corrections in the ratio M_K^2/M_π^2 according to eq. (3), involves the two effective coupling constants L_5 and L_8, which occur in the derivative expansion of the effective Lagrangian at first non-leading order [17]:

$$\Delta_M = \frac{8(M_K^2 - M_\pi^2)}{F_\pi^2}\,(2L_8 - L_5) + \chi\text{logs} \quad . \tag{7}$$

The term χlogs stands for the logarithms characteristic of chiral perturbation theory. They arise because the spectrum of the unperturbed Hamiltonian H_0 contains massless particles – the perturbation H_1 generates infra-red singularities. The coupling constant L_5 also determines the SU(3) asymmetry in the decay constants,

$$\frac{F_K}{F_\pi} = 1 + \frac{4(M_K^2 - M_\pi^2)}{F_\pi^2}\,L_5 + \chi\text{logs} \quad . \tag{8}$$

The comparison of eqs. (7) and (8) confirms that the symmetry breaking effects in the decay constants and in the mass spectrum are of similar nature. The calculation also reveals that the first order SU(3) correction in the mass ratio $(M_{K^0}^2 - M_{K^+}^2)/(M_K^2 - M_\pi^2)$ is the same as the one in M_K^2/M_π^2 [17]:

$$\frac{M_{K^0}^2 - M_{K^+}^2}{M_K^2 - M_\pi^2} = \frac{m_d - m_u}{m_s - \hat{m}}\{1 + \Delta_M + O(m^2)\} \quad . \tag{9}$$

Hence, the first order corrections drop out in the double ratio

$$Q^2 \equiv \frac{M_K^2}{M_\pi^2} \cdot \frac{M_K^2 - M_\pi^2}{M_{K^0}^2 - M_{K^+}^2} \quad . \tag{10}$$

The observed values of the meson masses thus provide a tight constraint on one particular ratio of quark masses:

$$Q^2 = \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}\{1 + O(m^2)\} \quad . \tag{11}$$

The constraint may be visualized by plotting the ratio m_s/m_d versus m_u/m_d [18]. Dropping the higher order contributions, the resulting curve takes the form of an ellipse:

$$\left(\frac{m_u}{m_d}\right)^2 + \frac{1}{Q^2}\left(\frac{m_s}{m_d}\right)^2 = 1 \quad , \tag{12}$$

with Q as major semi-axis (the term $\hat{m}^2/m_s^2$ has been discarded, as it is numerically very small).

6 Value of Q

The meson masses occurring in the double ratio (10) refer to pure QCD. The Dashen theorem states that in the chiral limit, the electromagnetic contributions to $M^2_{K^+}$ and to $M^2_{\pi^+}$ are the same, while the self energies of K^0 and π^0 vanish. Since the contribution to the mass difference between π^0 and π^+ from $m_d - m_u$ is negligibly small, the masses in pure QCD are approximately given by

$$\begin{aligned}&(M^2_{\pi^+})^{\rm QCD} = (M^2_{\pi^0})^{\rm QCD} = M^2_{\pi^0} \;\; ,\\ &(M^2_{K^+})^{\rm QCD} = M^2_{K^+} - M^2_{\pi^+} + M^2_{\pi^0} \;\; , \quad (M^2_{K^0})^{\rm QCD} = M^2_{K^0} \;\; ,\end{aligned}$$

where $M_{\pi^0}, M_{\pi^+}, M_{K^0}, M_{K^+}$ are the observed masses. Correcting for the electromagnetic self energies in this way, the quantity Q becomes

$$Q^2_D = \frac{(M^2_{K^0} + M^2_{K^+} - M^2_{\pi^+} + M^2_{\pi^0})(M^2_{K^0} + M^2_{K^+} - M^2_{\pi^+} - M^2_{\pi^0})}{4\, M^2_{\pi^0}\, (M^2_{K^0} - M^2_{K^+} + M^2_{\pi^+} - M^2_{\pi^0})} \;\; . \qquad (13)$$

Numerically, this yields $Q_D = 24.2$. The corresponding ellipse is shown in fig. 1 as a dash-dotted line. For this value of the semi-axis, the curve passes through the point specified by Weinberg's mass ratios, eq. (5).

The Dashen theorem is subject to corrections from higher order terms in the chiral expansion. As usual, there are two categories of contributions: loop graphs of order $e^2 m$ and terms of the same order from the derivative expansion of the effective e.m. Lagrangian. The Clebsch-Gordan coefficients occurring in the loop graphs are known to be large, indicating that two-particle intermediate states generate sizeable corrections; the corresponding chiral logarithms tend to increase the e.m. contribution to the kaon mass difference [19]. The numerical result depends on the scale used when evaluating the logarithms. In fact, taken by themselves, chiral logs are unsafe at any scale – one at the same time also needs to consider the contributions from the terms of order $e^2 m$ occurring in the effective Lagrangian. This is done in several recent papers [20]–[22], but the results are controversial. The authors of ref. [20] estimate the contributions arising from vector meson exchange and conclude that these give rise to large corrections, increasing the value $(M_{K^+} - M_{K^0})_{e.m.} = 1.3$ MeV predicted by Dashen to 2.3 MeV. According to ref. [21], however, the model used is in conflict with chiral symmetry: although the perturbations due to vector meson exchange are enhanced by a relatively

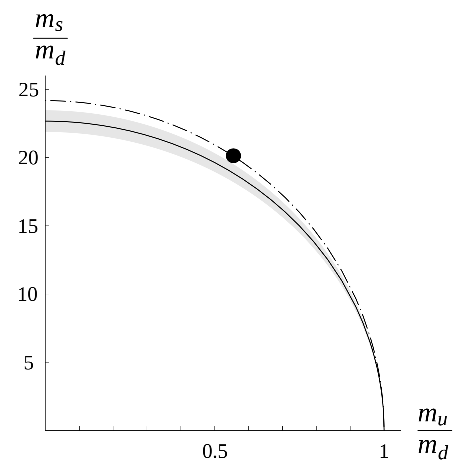

Figure 1: Elliptic constraint. The dot indicates Weinberg's mass ratios. The dash-dotted line represents the ellipse for the value $Q = 24.2$ of the semi-axis, obtained from the mass difference $K^0 - K^+$ with the Dashen theorem. The full line and the shaded region correspond to $Q = 22.7 \pm 0.8$, as required by the observed rate of the decay $\eta \to \pi^+\pi^-\pi^0$.

small energy denominator, chiral symmetry prevents them from being large. In view of this, it is puzzling that an evaluation based on the ENJL model yields an even larger effect, $(M_{K^+} - M_{K^0})_{e.m.} \simeq 2.6\,\text{MeV}$ [22]. More work is needed to clarify the situation. For the time being, the electromagnetic self energies of the kaons are subject to considerable uncertainties.[1]

The implications of the above estimates for the value of Q are illustrated on the rhs of fig. 2. The corresponding uncertainty in Q is rather modest, because the mass difference between K^+ and K^0 is predominantly due to $m_d > m_u$, not to the e.m. interaction: even if the Dashen theorem should underestimate the self energy by a factor of two, the corresponding prediction for Q only decreases by about 10 %.

The isospin-violating decay $\eta \to 3\pi$ allows one to measure the semi-axis in an entirely independent manner [24]. The transition amplitude is much less sensitive to the uncertainties associated with the electromagnetic interaction

[1]In the meantime, the electromagnetic self energies have been analysed within lattice QCD [23]. The result, $(M_{K^+} - M_{K^0})_{e.m.} = 1.9$ MeV, indicates that the corrections to the Dashen theorem are indeed substantial, although not quite as large as found in refs. [20, 22]. Expressed in terms of Q, the lattice result corresponds to $Q = 22.8$.

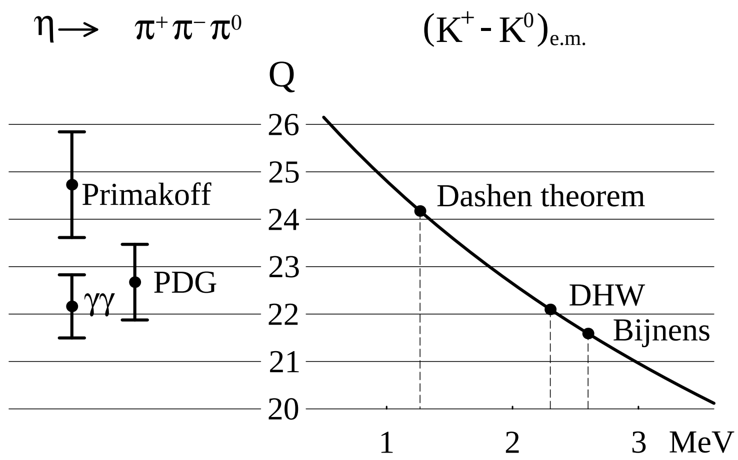

Figure 2: The lhs indicates the values of Q corresponding to the various experimental results for the rate of the decay $\eta\rightarrow\pi^+\pi^-\pi^0$. The rhs shows the results for Q obtained with three different theoretical estimates for the electromagnetic self energy of the kaons.

than the K^0-K^+ mass difference: the e.m. contribution is suppressed by chiral symmetry and is negligibly small [25]. The transition amplitude thus represents a sensitive probe of the symmetry breaking generated by m_d-m_u. To lowest order in the chiral expansion (current algebra), the amplitude of the transition $\eta\rightarrow\pi^+\pi^-\pi^0$ is given by

$$A=-\frac{\sqrt{3}}{4}\,\frac{m_d-m_u}{m_s-\hat{m}}\,\frac{1}{F_\pi^2}\,(s-\tfrac{4}{3}M_\pi^2)\;\;,$$

where s is the square of the centre-of-mass energy of the charged pion pair. The corrections of first non-leading order (chiral perturbation theory to one loop) are also known. It is convenient to write the decay rate in the form $\Gamma_{\eta\rightarrow\pi^+\pi^-\pi^0}=\Gamma_0\,(Q_D/Q)^4$, where Q_D is specified in eq. (13). As shown in ref. [24], the one loop calculation yields a parameter free prediction for the constant Γ_0. Updating the value of F_π, the numerical result reads $\Gamma_0=168\pm50\,\mathrm{eV}$. Although the calculation includes all corrections of first non-leading order, the error bar is large. The problem originates in the final state interaction, which strongly amplifies the transition probability in part of the Dalitz plot. The one loop calculation does account for this phenomenon, but only to leading order in the low energy expansion. The final state interaction is analysed more accurately in two recent papers [26, 27], which exploit the fact that analyticity and unitarity determine the amplitude up to a few subtraction constants. For these, the corrections to the current algebra predictions are small, because they are barely affected by the final state interaction. Although the dispersive framework used in the two papers differs, the results are nearly the same: While Kambor, Wiesendanger and Wyler obtain $\Gamma_0 = 209 \pm 20\,\mathrm{eV}$, we get $\Gamma_0 = 219 \pm 22\,\mathrm{eV}$. This shows that the theoretical uncertainties of the dispersive calculation are small. Since the decay rate is proportional to Q^{-4}, the transition $\eta \to 3\pi$ represents an extremely sensitive probe, allowing a determination of Q to an accuracy of about $2\frac{1}{2}\%$.

Unfortunately, however, the experimental situation is not clear [28]. The value of $\Gamma_{\eta\to\pi^+\pi^-\pi^0}$ relies on the rate of the decay into two photons. The two different methods of measuring $\Gamma_{\eta\to\gamma\gamma}$ – photon-photon-collisions and Primakoff effect – yield conflicting results. While the data based on the Primakoff effect are in perfect agreement with the number $Q = 24.2$ which follows from the Dashen theorem, the $\gamma\gamma$ data yield a significantly lower result (see lhs of fig. 2). The statistics is dominated by the $\gamma\gamma$ data. Using the overall fit of the Particle Data Group, $\Gamma_{\eta\to\pi^+\pi^-\pi^0} = 283 \pm 28\,\mathrm{eV}$ [28], and adding errors quadratically, we obtain $Q = 22.7 \pm 0.8$, to be compared with the value $Q = 22.4 \pm 0.9$ given in ref. [26]. The result appears to confirm the conclusions reached in [20, 22]. The above discussion makes it clear that an improvement of the experimental situation concerning $\Gamma_{\eta\to\gamma\gamma}$ is of considerable interest.

7 A phenomenological ambiguity

Chiral perturbation theory thus fixes one of the two quark mass ratios in terms of the other, to within small uncertainties. The ratios themselves, i.e. the position on the ellipse, are a more subtle issue. Kaplan and Manohar [18] pointed out that the corrections to the lowest order result, eq. (5), cannot be determined on purely phenomenological grounds. They argued that these corrections might be large and that the u-quark might actually be massless. This possibility is widely discussed in the literature [29], because the strong CP problem would then disappear.

The reason why phenomenology alone does not allow us to determine the two individual ratios beyond leading order is the following. The matrix

$$m' = \alpha_1 m + \alpha_2 (m^+)^{-1} \det m$$

transforms in the same manner as m. For a real, diagonal mass matrix, the transformation amounts to

$$m'_u = \alpha_1 m_u + \alpha_2 m_d m_s \quad (\text{cycl. } u \to d \to s \to u) \ . \tag{14}$$

Symmetry does therefore not distinguish m' from m. Since the effective theory exclusively exploits the symmetry properties of QCD, the above transformation of the quark mass matrix does not change the form of the effective Lagrangian – the transformation may be absorbed in a suitable change of the effective coupling constants [18]. This implies, however, that the expressions for the masses of the pseudoscalars, as well as for the scattering amplitudes or for the matrix elements of the vector and axial currents, which follow from this Lagrangian, are invariant under the operation $m \to m'$. Conversely, the experimental information on the various observables does not allow one to distinguish m from m'. Indeed, one readily checks that the transformation $m \to m'$ maps the ellipse onto itself (up to terms of order $(m_u - m_d)^2/m_s^2$, which were neglected). Since the position on the ellipse does not remain invariant, it cannot be extracted from these observables within chiral perturbation theory.

One is not dealing with a hidden symmetry of QCD here – this theory is not invariant under the change (14) of the quark masses. In particular, the matrix elements of the scalar and pseudoscalar operators are modified. The Ward identity for the axial current implies, for example, that the vacuum-to-pion matrix element of the pseudoscalar density is given by

$$\langle 0|\bar{d}\, i\gamma_5 u|\pi^+\rangle = \sqrt{2}\, F_\pi M_{\pi^+}^2/(m_u + m_d) \ . \tag{15}$$

The relation is exact, except for electroweak corrections. It involves the physical quark masses and is not invariant under the above transformation. Unfortunately, however, an experimental probe sensitive to the scalar or pseudoscalar currents does not exist – the electromagnetic and weak interactions happen to probe the low energy structure of the system exclusively through vector and axial currents.

8 Estimates and bounds

I now discuss the size of the corrections to the leading order formulae (4) for the two quark mass ratios m_u/m_d and m_s/m_d. For the reasons just described, this discussion necessarily involves a theoretical input of one sort or another. To clearly identify the relevant ingredient, I explicitly formulate it as hypothesis A, B, ...

Hypothesis A : Assume that the corrections of order m^2 or higher are small and neglect these.

This is the attitude taken in early work on the problem [1]. In the notation used above, the assumption amounts to $\Delta_M \simeq 0$, so that $m_s/\hat{m} \simeq (2M_K^2 - M_\pi^2)/M_\pi^2 \simeq 26$. In the plane spanned by m_u/m_d and m_s/m_d, this represents a straight line. The intersection with the ellipse then fixes things. It is convenient to parametrize the position on the ellipse by means of the ratio R, which measures the relative size of isospin and SU(3) breaking,

$$R \equiv \frac{m_s - \hat{m}}{m_d - m_u} \quad . \tag{16}$$

With the value $Q = 24.2$ (Dashen theorem), the intersection occurs at the mass ratios given by Weinberg, which correspond to $R \simeq 43$. For the value of the semi-axis which follows from η decay, $Q = 22.7$, the intersection instead takes place at $R \simeq 39$.

The baryon octet offers a good test: Applying the hypothesis to the chiral expansion of the baryon masses, i.e. disregarding terms of order m^2, one arrives at three independent estimates for R, viz. 51 ± 10 $(N - P)$, 43 ± 4 $(\Sigma^- - \Sigma^+)$ and 42 ± 6 $(\Xi^- - \Xi^0)$.[2] Within the errors, these results are consistent with the values $R \simeq 43$ and 39, obtained above from $K^0 - K^+$ and from $\eta \to \pi^+\pi^-\pi^0$, respectively. A recent reanalysis of $\rho - \omega$ mixing [30] leads to $R = 41 \pm 4$ and thus corroborates the picture further.

Another source of information concerning the ratio of isospin and SU(3) breaking effects is the branching ratio $\Gamma_{\psi' \to \psi\pi^0}/\Gamma_{\psi' \to \psi\eta}$. The chiral expansion

[2] Note that, in this case, the expansion contains terms of order $m^{\frac{3}{2}}$, which do play a significant role numerically. The error bars represent simple rule-of-thumb estimates, indicated by the noise visible in the calculation. For details see ref. [1].

of the corresponding ratio of transition amplitudes starts with [31]:

$$\frac{\langle\psi\pi^0|\bar{q}mq|\psi'\rangle}{\langle\psi\eta|\bar{q}mq|\psi'\rangle} = \frac{3\sqrt{3}}{4\,R}\{1+\Delta_{\psi'}+\ldots\} \ .$$

Disregarding the correction $\Delta_{\psi'}$, which is of order $m_s-\hat{m}$, the data imply $R=31\pm 4$, where the error bar corresponds to the experimental accuracy of the branching ratio. The value is significantly lower than those listed above. The higher order corrections are discussed in ref. [32], but the validity of the multipole expansion used there is questionable [33]. The calculation is of interest, because it is independent of other determinations, but at the present level of theoretical understanding, it is subject to considerable uncertainties. Since the quark mass ratios given in refs. [34] rely on the value of R obtained in this way, they are subject to the same reservations. Nevertheless, the information extracted from ψ' decays is useful, because it puts an upper limit on the value of R. As an SU(3) breaking effect, the correction $\Delta_{\psi'}$ is expected to be of order 25%. The estimate $|\Delta_{\psi'}|<0.4$ is on the conservative side. Expressed in terms of R, this implies $R<44$.

Hypothesis B : Assume that the effective coupling constants are dominated by the singularities which are closest to the origin.

This amounts to a generalization of the vector meson dominance hypothesis and yields rough estimates for the various coupling constants, e.g. [35]

$$L_5 \simeq \frac{F_\pi^2}{4\,M_{a_0}^2} \ , \quad L_7 \simeq -\frac{F_\pi^2}{48\,M_{\eta'}^2} \ , \quad L_9 \simeq \frac{F_\pi^2}{2\,M_\rho^2} \ , \quad \ldots$$

In all cases where direct phenomenological information is available, these estimates do remarkably well. Also, this framework explains why it is justified to treat m_s as a perturbation [36]: at order p^4, the symmetry breaking part of the effective Lagrangian is determined by the constants $L_4,\ldots,L_8$. These are immune to the low energy singularities generated by spin 1 resonances, but are affected by the exchange of scalar or pseudoscalar particles. Their magnitude is therefore determined by the scale $M_{a_0}\simeq M_{\eta'}\simeq 1$ GeV. According to eq. (8), the asymmetry in the decay constants, for instance, is given by

$$\frac{F_K}{F_\pi} = 1+\frac{M_K^2-M_\pi^2}{M_{a_0}^2}+\chi\text{logs} \ . \qquad (17)$$

This shows that the breaking of the chiral and eightfold way symmetries is controlled by the mass ratio of the Goldstone bosons to the non-Goldstone states of spin zero, $M_K^2/M_{a_0}^2 \simeq M_K^2/M_{\eta'}^2 \simeq \frac{1}{4}$. In chiral perturbation theory, the observation that the Goldstones are the lightest hadrons thus acquires quantitative significance.

The above estimates in particular also imply that the correction Δ_M is small: B is consistent with A.

Hypothesis C : Assume that the large N_c expansion makes sense for $N_c = 3$.

As noted already in ref. [37], the ambiguity discussed in section 7 disappears in the large-N_c limit, because the Kaplan-Manohar transformation violates the Zweig rule. In this limit, the quark loop graph that gives rise to the anomaly in the divergence of the singlet axial current is suppressed, so that QCD acquires an additional U(1) symmetry, whose spontaneous breakdown gives rise to a ninth Goldstone boson, the η' [38]–[40]. The implications for the effective Lagrangian are extensively discussed in the literature [41] and the leading terms in the expansion in powers of $1/N_c$ have been worked out. More recently, the analysis was extended to first non-leading order, accounting for all terms which are suppressed either by one power of $1/N_c$ or by one power of the quark mass matrix [42]. This framework leads to a bound for Δ_M, which arises as follows.

At leading order of the chiral expansion, the mass of the η is given by the Gell-Mann-Okubo formula. There are two categories of corrections of first non-leading order: (i) The first is of the same origin as the correction which occurs in the mass formula (3) for the ratio M_K^2/M_π^2 and is also determined by Δ_M: the expression for the mass of the η, which follows from the Gell-Mann-Okubo formula, $M_\eta^2 = \frac{1}{3}(4M_K^2 - M_\pi^2)$, is replaced by

$$m_1^2 = \tfrac{1}{3}(4M_K^2 - M_\pi^2) + \tfrac{4}{3}(M_K^2 - M_\pi^2)\,\Delta_M \quad .$$

(ii) In addition, there is mixing between the two states η, η'. The levels repel in proportion to the square of the transition matrix element $\sigma_1 \propto \langle \eta' | \bar{q} m q | \eta \rangle$, so that the mass formula for the η takes the form

$$M_\eta^2 = m_1^2 - \frac{\sigma_1^2}{M_{\eta'}^2 - m_1^2} \quad . \tag{18}$$

This immediately implies the inequality $M_\eta^2 < m_1^2$, i.e.

$$\Delta_M > -\frac{4M_K^2 - 3M_\eta^2 - M_\pi^2}{4(M_K^2 - M_\pi^2)} = -0.07 \ .$$

At leading order of the expansion, the transition matrix element σ_1 is given by $\sigma_0 = \frac{2}{3}\sqrt{2}\,(M_K^2 - M_\pi^2)$. There are again two corrections of first non-leading order: $\sigma_1 = \sigma_0\,(1 + \Delta_M - \Delta_N)$. The first is an SU(3) breaking effect of order $m_s - \hat{m}$, determined by Δ_M, while Δ_N represents a correction of order $1/N_c$ of unknown size – the mass formula (18) merely fixes Δ_N as a function of Δ_M or vice versa. A coherent picture, however, only results if both $|\Delta_M|$ and $|\Delta_N|$ are small compared with unity. If the above inequality were saturated, σ_1 would have to vanish, i.e. $1 + \Delta_N - \Delta_M = 0$. In other words, the corrections would have to cancel the leading term. It is clear that, in such a situation, the expansion is out of control. Accordingly, Δ_M must be somewhat larger than -0.07. As Δ_M grows, Δ_N decreases. Even $\Delta_M = 0$ calls for large Zweig rule violations, $\Delta_N \simeq \frac{1}{2}$. The condition

$$\Delta_M > 0 \tag{19}$$

thus represents a generous lower bound for the region where a truncated $1/N_c$ expansion leads to meaningful results. It states that the current algebra formula, which relates the quark mass ratio $m_s/\hat{m}$ to the meson mass ratio M_K^2/M_π^2, represents an upper limit, $m_s/\hat{m} < 2M_K^2/M_\pi^2 - 1 = 25.9$.

This shows that A, B and C are mutually consistent, provided Δ_M is small and positive. The bound (19) is shown in fig. 3: mass ratios in the hatched region are in conflict with the hypothesis that the first two terms of the $1/N_c$ expansion yield meaningful results for $N_c = 3$. Since the Weinberg ratios correspond to $\Delta_M = 0$, they are located at the boundary of this region. In view of the elliptic constraint, the bound in particular implies $m_u/m_d \gtrsim \frac{1}{2}$.

Hypothesis D : Assume that m_u vanishes.

It is clear that this assumption violates the large N_c bound just discussed. D is also inconsistent with A and B. In fact, as pointed out in refs. [43], this hypothesis leads to a very queer picture, for the following reason.

The lowest order mass formulae (2) and (3) imply that the ratio m_u/m_d determines the K^0/K^+ mass difference, the scale being set by M_π:

$$M_{K^0}^2 - M_{K^+}^2 = \frac{m_d - m_u}{m_u + m_d}\, M_\pi^2 + \dots$$

The formula holds up to corrections from higher order terms in the chiral expansion and up to e.m. contributions. Setting $m_u = 0$, the relation predicts $M_{K^0} - M_{K^+} \simeq 16\,\mathrm{MeV}$, four times larger than the observed mass difference. The disaster can only be blamed on the higher order terms, because the electromagnetic self energies are much too small. Under such circumstances, it does not make sense to truncate the expansion at first non-leading order. The conclusion to be drawn from the assumption $m_u = 0$ is that chiral perturbation theory is unable to account for the masses of the Goldstone bosons. It is difficult to understand how a framework with a basic flaw like this can be so successful.

The assumption $m_u = 0$ also implies that the matrix elements of the scalar and pseudoscalar currents must exhibit very strong SU(3) breaking effects [43]. Consider e.g. the pion and kaon matrix elements of the scalar operators $\bar{u}u, \bar{d}d, \bar{s}s$. In the limit $m_d = m_s$, the ratio

$$r = \frac{\langle \pi^+ | \bar{u}u - \bar{s}s | \pi^+ \rangle}{\langle K^+ | \bar{u}u - \bar{d}d | K^+ \rangle}$$

is equal to 1. The SU(3) breaking effects are readily calculated by working out the derivatives of $M^2_{\pi^+}, M^2_{K^+}$ with respect to m_u, m_d, m_s. Neglecting the chiral logarithms which turn out to be small in this case, the first order corrections may be expressed in terms of the masses,

$$r = \left(\frac{m_s - m_u}{m_d - m_u} \cdot \frac{M^2_{K^0} - M^2_{\pi^+}}{M^2_{K^0} - M^2_{K^+}} \right)^2 \left\{ 1 + O(m^2) \right\} \quad .$$

The relation is of the same character as the one that leads to the elliptic constraint: the corrections are of second order in the quark masses. For $m_u = 0$, the elliptic constraint reduces to $m_s/m_d = Q + \frac{1}{2}$, so that the relation predicts $r \simeq 3$, the precise value depending on the number used for the electromagnetic contribution to $M_{K^+} - M_{K^0}$. So, $m_u = 0$ leads to the prediction that the evaluation of the above matrix elements with sum rule or lattice techniques will reveal extraordinarily strong flavour symmetry breaking effects – a bizarre picture. For me this is enough to stop talking about $m_u = 0$ here.

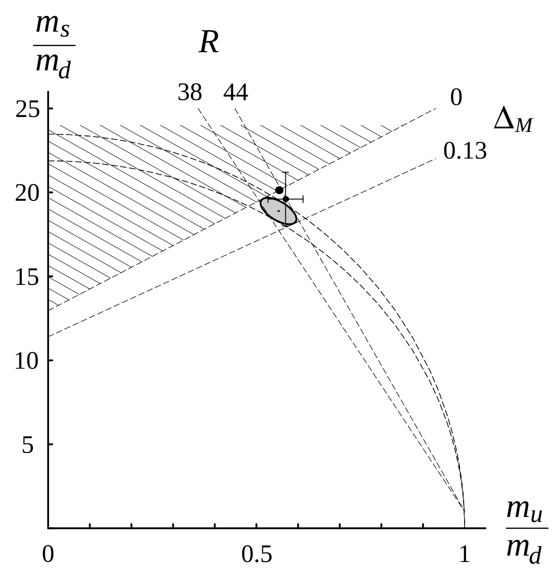

Figure 3: Quark mass ratios. The dot corresponds to Weinberg's values, while the cross represents the estimates given in ref. [1]. The hatched region is excluded by the bound $\Delta_M > 0$. The error ellipse shown is characterized by the constraints $Q = 22.7 \pm 0.8$, $\Delta_M > 0$, $R < 44$, which are indicated by dashed lines.

9 Conclusions

1. The mass of the strange quark is known quite accurately from QCD sum rules:

$$m_s = 175 \pm 25\,\mathrm{MeV} \quad (\overline{\mathrm{MS}}\text{ scheme at } \mu = 1\,\mathrm{GeV}) \ .$$

2. The ratios m_u/m_d and m_s/m_d are constrained to an ellipse, whose small semi-axis is equal to 1,

$$\left(\frac{m_u}{m_d}\right)^2 + \frac{1}{Q^2}\left(\frac{m_s}{m_d}\right)^2 = 1 \ .$$

η decay yields a remarkably precise measurement of the large semi-axis,

$$Q = 22.7 \pm 0.8 \ .$$

Unfortunately, however, the experimental situation concerning the lifetime of the η is not satisfactory – the given error bar relies on the averaging procedure used by the Particle Data Group.

3. The position on the ellipse cannot accurately be determined from phenomenology alone. The theoretical arguments given imply that the corrections to Weinberg's leading order mass formulae are small. In particular, there is a new bound based on the $1/N_c$ expansion, which requires $m_u/m_d \gtrsim \frac{1}{2}$ and thereby eliminates the possibility that the u-quark is massless.

4. The final result for the quark mass ratios is indicated by the shaded error ellipse in fig. 3, which is defined by the following three constraints: (i) On the upper and lower sides, the ellipse is bounded by the two dashed lines that correspond to $Q = 22.7 \pm 0.8$. (ii) To the left, it touches the hatched region, excluded by the large-N_c bound. (iii) On the right, I use the upper limit $R < 44$, which follows from the observed value of the branching ratio $\Gamma_{\psi' \to \psi \pi^0}/\Gamma_{\psi' \to \psi \eta}$. The corresponding range of the various parameters of interest is

$$\frac{m_u}{m_d} = 0.553 \pm 0.043 \;, \quad \frac{m_s}{m_d} = 18.9 \pm 0.8 \;, \quad \frac{m_s}{m_u} = 34.4 \pm 3.7 \;,$$
$$\frac{m_s - \hat{m}}{m_d - m_u} = 40.8 \pm 3.2 \;, \quad \frac{m_s}{\hat{m}} = 24.4 \pm 1.5 \;, \quad \Delta_M = 0.065 \pm 0.065 \;.$$

While the central value for m_u/m_d happens to coincide with the leading order formula, the one for m_s/m_d turns out to be slightly smaller. The difference, which amounts to 6%, originates in the fact that the available data on the η lifetime imply a somewhat smaller value of Q than what is predicted by the Dashen theorem.

5. The theoretical arguments discussed as hypotheses A, B and C in section 8 are perfectly consistent with these numbers. In particular, the early determinations of R, based on the baryon mass splittings and on ρ–ω mixing [1], are confirmed. The rough estimate $m_s/\hat{m} = 29 \pm 7$, obtained by Bijnens, Prades and de Rafael from QCD sum rules [5], provides an independent check: The lower end of this interval corresponds to $\Delta_M < 0.17$. Fig. 3 shows that this constraint restricts the allowed region to the right and is only slightly weaker than the condition $R < 44$ used above.

6. Together with the value of m_s, the ratios finally also determine the size of m_u and m_d:

$$m_u = 5.1 \pm 0.9\,\mathrm{MeV} \;, \quad m_d = 9.3 \pm 1.4\,\mathrm{MeV} \;.$$

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